IFT-P.041/2000

Consistency of Super-Poincaré Covariant Superstring Tree Amplitudes

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Using pure spinors, the superstring was recently quantized in a manifestly ten-dimensional super-Poincaré covariant manner and a covariant prescription was given for tree-level scattering amplitudes. In this paper, we prove that this prescription is cyclically symmetric and, for the scattering of an arbitrary number of massless bosons and up to four massless fermions, it agrees with the standard Ramond-Neveu-Schwarz prescription.

April 2000

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1. Introduction

In a recent paper [1], a new formalism was proposed for quantizing the superstring in a manifestly ten-dimensional super-Poincaré covariant manner. Unlike all previous such proposals, an explicit covariant prescription was given for computing tree-level scattering amplitudes of an arbitrary number of states. To check consistency of the formalism, one would obviously like to prove that this new prescription for tree amplitudes is equivalent to the standard Ramond-Neveu-Schwarz (RNS) prescription[2].

In this paper, we prove this equivalence for tree amplitudes involving an arbitrary number of massless bosons and up to four massless fermions. We do not yet have an equivalence proof for amplitudes involving massive states or more than four massless fermions, however, we suspect it might be possible to construct such a proof using factorization arguments together with the results of this paper.

After reviewing the super-Poincaré covariant formalism in section 2, we prove in section 3 that the covariant amplitude prescription for tree amplitudes is cyclically symmetric, i.e. it does not depend on which three of the vertex operators are chosen to be unintegrated. The proof of cyclic symmetry is not the standard one since there is no natural b ghost in the covariant formalism. In section 4, we prove by explicit analysis that the covariant and RNS prescriptions are equivalent for tree amplitudes involving an arbitrary number of massless bosons and four massless fermions. In section 5, we similarly prove equivalence for tree amplitudes involving an arbitrary number of massless bosons and two massless fermions. And in section 6, we use supersymmetry together with the results of section 5 to prove equivalence for tree amplitudes involving an arbitrary number of massless bosons and zero fermions.

2. Review of Super-Poincaré Covariant Formalism

The worldsheet variables in the new formalism include the usual ten-dimensional superspace variables x^m and θ^α ($m = 0$ to 9 and $\alpha = 1$ to 16), as well as a bosonic spinor variable λ^α satisfying the pure spinor constraint $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$ for $m = 0$ to 9.[3] Although one can solve the pure spinor constraint in terms of independent variables as in [1], this will not be necessary for computing scattering amplitudes. In addition to the above worldsheet

[3] $\gamma^m_{\alpha\beta}$ and $\gamma^{m\,\alpha\beta}$ are 16×16 symmetric matrices which are the off-diagonal elements of the 32×32 gamma-matrices and which satisfy $\gamma^m_{\alpha\beta}\gamma^{n\,\beta\gamma} + \gamma^n_{\alpha\beta}\gamma^{m\,\beta\gamma} = 2\eta^{mn}\delta^\gamma_\alpha$ and $\gamma_{m\,(\alpha\beta}\gamma^m_{\gamma)\delta} = 0$.

spin-zero variables, the formalism also contains the worldsheet spin-one variables p_α and N^{mn}, which are respectively the conjugate momentum to θ^α and the Lorentz currents for λ^α.

The OPE's of these worldsheet variables are

$$p_\alpha(y)\theta^\beta(z) \to \frac{\delta_\alpha^\beta}{y-z}, \quad x^m(y)x^n(z) \to -\eta^{mn}\log|y-z|, \tag{2.1}$$

$$N^{mn}(y)\lambda^\alpha(z) \to \frac{(\gamma^{mn})^\alpha{}_\beta \lambda^\beta(z)}{2(y-z)},$$

$$N^{kl}(y)N^{mn}(z) \to \frac{\eta^{m[l}N^{k]n}(z) - \eta^{n[l}N^{k]m}(z)}{y-z} - 3\frac{\eta^{kn}\eta^{lm} - \eta^{km}\eta^{ln}}{(y-z)^2}, \tag{2.2}$$

where $(\gamma^{mn})^\alpha{}_\beta = -(\gamma^{mn})_\beta{}^\alpha = \frac{1}{2}(\gamma^{m\,\alpha\gamma}\gamma^n_{\gamma\beta} - \gamma^{n\,\alpha\gamma}\gamma^m_{\gamma\beta})$. As in [3], it is convenient to define the combinations

$$d_\alpha = p_\alpha - \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta\partial x^m - \frac{1}{8}\gamma^m_{\alpha\beta}\gamma_{m\,\gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta, \quad \Pi^m = \partial x^m + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\alpha\partial\theta^\beta \tag{2.3}$$

which satisfy the OPE's

$$d_\alpha(y)d_\beta(z) \to -\frac{\gamma^m_{\alpha\beta}\Pi_m(z)}{y-z}, \quad d_\alpha(y)\Pi^m(z) \to \frac{\gamma^m_{\alpha\beta}\partial\theta^\beta(z)}{y-z}, \tag{2.4}$$

and which commute with the spacetime-supersymmetry generator[4]

$$q_\alpha = -\oint dz(p_\alpha + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta\partial x^m + \frac{1}{24}\gamma^m_{\alpha\beta}\gamma_{m\,\gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta). \tag{2.5}$$

Physical vertex operators in the super-Poincaré covariant formalism are defined to be in the cohomology of the BRST-like operator $Q = \oint dz\lambda^\alpha d_\alpha$. As discussed in [1], the unintegrated massless vertex operator for the open superstring is $U = \lambda^\alpha A_\alpha(x,\theta)$ where $A_\alpha(x,\theta)$ is the spinor potential for super-Yang-Mills satisfying $D_\alpha(\gamma^{mnpqr})^{\alpha\beta}A_\beta = 0$ for any five-form $mnpqr$ and $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta\partial_m$. The spinor potential is defined up to a gauge transformation $A_\alpha \to A_\alpha + D_\alpha\Omega$, which allows one to choose the gauge

$$A_\alpha(x,\theta) = \frac{1}{2}a_m(x)\gamma^m_{\alpha\beta}\theta^\beta + \frac{1}{3}\xi^\gamma(x)\gamma^m_{\alpha\beta}\gamma_{m\,\gamma\delta}\theta^\beta\theta^\delta + ... \tag{2.6}$$

where $a_m(x)$ and $\xi^\alpha(x)$ are the linearized on-shell gluon and gluino of super-Yang-Mills satisfying $\partial^m\partial_m a_n = \partial^n a_n = \gamma^m_{\alpha\beta}\partial_m\xi^\beta = 0$ and ... denotes terms higher order in θ which

[4] We have chosen conventions such that $\{q_\alpha, q_\beta\} = \gamma^m_{\alpha\beta}\oint dz\partial x_m$ to simplify the comparison with RNS amplitudes.

depend on derivatives of a_m and ξ^α. So in the gauge of (2.6), the unintegrated gluon and gluino vertex operators are

$$U_m^B = [\frac{1}{2}\lambda\gamma_m\theta + ...]e^{ik\cdot x}, \quad U_\alpha^F = [\frac{1}{3}(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha + ...]e^{ik\cdot x}. \tag{2.7}$$

To compute scattering amplitudes, one also needs to define vertex operators in integrated form. Although there is no natural b ghost in this formalism, one can define the integrated vertex operator for a physical state, $U = \int dzV$, by requiring that $[Q, V] = \partial U$ where U is the unintegrated vertex operator [4]. For the massless states, $V = \Pi^m A_m + \partial\theta^\alpha A_\alpha + d_\alpha W^\alpha + \frac{1}{2}N^{mn}F_{mn}$ where $A_m = \frac{1}{8}\gamma_m^{\alpha\beta}D_\alpha A_\beta$ is the vector potential, $W^\alpha = \frac{1}{10}\gamma^{m\,\alpha\beta}(D_\beta A_m - \partial_m A_\beta)$ is the spinor field strength, and $F_{mn} = \partial_{[m}A_{n]} = \frac{1}{8}(\gamma_{mn})^\beta{}_\alpha D_\beta W^\alpha$ is the vector field strength. So in the gauge of (2.6), the integrated gluon and gluino vertex operators are

$$V_m^B = [\partial x_m - ik^n(N_{mn} - \frac{1}{2}p\gamma_{mn}\theta) + ...]e^{ik\cdot x}, \quad V_\alpha^F = -[p_\alpha + ...]e^{ik\cdot x}, \tag{2.8}$$

where the term proportional to $p\gamma_{mn}\theta$ in V_m^B comes from the $d_\alpha W^\alpha$ term in V. As will be shown later, the higher-order θ terms denoted by ... in (2.7) and (2.8) will not contribute to tree-level scattering amplitudes involving up to four fermions.

The N-point tree-level scattering amplitude is defined by taking the worldsheet correlation function of three unintegrated vertex operators and $N-3$ integrated vertex operators, i.e.

$$\mathcal{A} = \langle U_1(z_1)U_2(z_2)U_3(z_3)\int dz_4 V_4(z_4)...\int dz_N V_N(z_N)\rangle. \tag{2.9}$$

The only subtle point in computing this correlation function comes from the zero modes of λ^α and θ^α. The correlation function over these zero modes is defined to vanish unless one has three λ zero modes and five θ zero modes contracted in the combination $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$. More explicitly, after performing the correlation function over x^m and over the non-zero modes of θ^α and λ^α, the amplitude is obtained by defining

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 2880 \tag{2.10}$$

where the normalization factor of 2880 has been chosen to give agreement with the RNS normalization. This is equivalent to defining the correlation function over the zero modes of $Y(x, \theta, \lambda)$ to be proportional to

$$\int d^{10}x \int d\Omega(\bar\lambda_\rho\lambda^\rho)^{-3}\gamma_{mnp}^{\alpha\beta}(\bar\lambda\gamma^m)^\gamma(\bar\lambda\gamma^n)^\delta(\bar\lambda\gamma^p)^\kappa \int d\theta_\alpha d\theta_\beta d\theta_\gamma d\theta_\delta d\theta_\kappa Y, \tag{2.11}$$

where $\bar{\lambda}_\alpha$ is the complex conjugate of λ^α (after Wick-rotating to Euclidean space) and $d\Omega$ is an integration over the different possible orientations of λ^α. (2.11) can be interpreted as integration over an on-shell harmonic superspace since, as was shown in [1], it preserves spacetime-supersymmetry and gauge invariance. Note that integration over all sixteen θ's leads to inconsistencies as was noted in [5].

3. Cyclic Symmetry of Tree Amplitudes

The amplitude prescription of (2.9) fixes three of the vertex operators to be unintegrated and the remaining vertex operators to be integrated. The choice of which three vertex operators are unintegrated breaks the manifest cyclic symmetry of the computation, i.e. the symmetry under a cyclic permutation of the external states. To show that the resulting amplitude is indeed cyclically symmetric, one therefore needs to prove that the prescription is independent of which three vertex operators are chosen to be unintegrated.

In the RNS (or bosonic string) amplitude prescription, the independence of the choice of which three vertex operators are unintegrated can be proven using manipulations of the b ghost[6]. This follows from the fact that the integrated vertex operator $\int dzV$ is related to the unintegrated vertex operator U by $V = \{b, U\}$. In the super-Poincaré covariant formalism, there is no natural candidate for the b ghost so such a proof cannot be used.

Alhough one cannot use that $V = \{b, U\}$ in the covariant formalism, one can use that $[Q, V] = \partial U$ [4]. Note that $[Q, V] = \partial U$ is also satisfied in the RNS and bosonic string, so the proof in this section serves as an alternative to the conventional proof using manipulations of the b ghost. Our proof will argue that

$$\langle U_1(z_1)U_2(z_2)U_3(z_3)\int_{z_3}^{z_1} dz_4V_4(z_4)\int_{z_4}^{z_1} dz_5V_5(z_5)...\int_{z_{N-1}}^{z_1} dz_NV_N(z_N)\rangle = \tag{3.1}$$

$$\langle U_1(z_1)U_2(z_2)\int_{z_2}^{z_3} dyV_3(y)\ U_4(z_3)\int_{z_3}^{z_1} dz_5V_5(z_5)\int_{z_5}^{z_1} dz_6V_6(z_6)...\int_{z_{N-1}}^{z_1} dz_NV_N(z_N)\rangle$$

where $z_1 < z_2 < ... < z_N$ and the integration with upper limit z_1 signifies an integration on the compactified real line which includes the point at ∞. Similar arguments can be used to prove equivalence of the amplitude prescription for any choice of the three unintegrated vertex operators.

To prove (3.1), first write the left-hand side of (3.1) as

$$\langle U_1(z_1)U_2(z_2)\int_{z_2}^{z_3} dy[Q, V_3(y)]\int_{z_3}^{z_1} dz_4V_4(z_4)\int_{z_4}^{z_1} dz_5V_5(z_5)...\int_{z_{N-1}}^{z_1} dz_NV_N(z_N)\rangle \tag{3.2}$$

where we have used that $\int_{z_2}^{z_3} dy[Q, V_3(y)] = U_3(z_3) - U_3(z_2)$. The contribution coming from $U_3(z_2)$ can be ignored since when $k_2 \cdot k_3$ is sufficiently large, $U_2(z_2)U_3(z_2+\epsilon) \to \epsilon^{k_2 \cdot k_3} \to 0$ as $\epsilon \to 0$. But since the amplitude is analytic (except for poles) in the momentum, the contribution coming from $U_3(z_2)$ must vanish for all k_2 and k_3 if it vanishes for some region of k_2 and k_3. This is the 'cancelled propagator' argument discussed in [6].

Using properties of the correlation function discussed in [1], one can pull the BRST operator off of $V_3(z_3)$ until it circles either $V_4(z_4)$, $V_5(z_5)$, ... , $V_N(z_N)$. The contribution coming from when it circles $V_4(z_4)$ is

$$\langle U_1(z_1)U_2(z_2) \int_{z_2}^{z_3} dy V_3(y) \int_{z_3}^{z_1} dz_4[-Q, V_4(z_4)] \int_{z_4}^{z_1} dz_5 V_5(z_5) ... \int_{z_{N-1}}^{z_1} dz_N V_N(z_N)\rangle =$$

$$\langle U_1(z_1)U_2(z_2) \int_{z_2}^{z_3} dy V_3(y)\ U_4(z_3) \int_{z_3}^{z_1} dz_5 V_5(z_5) \int_{z_5}^{z_1} dz_6 V_6(z_6) ... \int_{z_{N-1}}^{z_1} dz_N V_N(z_N)\rangle \tag{3.3}$$

where the contribution from $U_4(z_1)$ in (3.3) has been ignored using the cancelled propagator argument described above. Similarly, the contributions from Q circling any of $V_5(z_5)...V_N(z_N)$ can be ignored since they only give rise to terms which vanish due to the cancelled propagator argument. Since (3.3) is equal to the right-hand side of (3.1), we have proven our claim.

Similar methods can be used to prove that closed superstring tree amplitudes are independent of the choice of which three vertex operators are unintegrated. For the closed superstring, the unintegrated vertex operator $U(z, \bar{z})$ is related to the integrated vertex operator $\int d^2z V(z, \bar{z})$ by

$$\{Q[\bar{Q}, V]\} = \partial\bar{\partial}U \tag{3.4}$$

where Q and $\bar{Q}$ are the holomorphic and anti-holomorphic BRST operators. So the closed superstring tree amplitude

$$\mathcal{A} = \langle U_1(z_1, \bar{z}_1)U_2(z_2, \bar{z}_2)U_3(z_3, \bar{z}_3) \int d^2z_4 V_4(z_4, \bar{z}_4) ... \int d^2z_N V_N(z_N, \bar{z}_N)\rangle \tag{3.5}$$

can be written as

$$\mathcal{A} = \frac{1}{2\pi}\langle U_1(z_1, \bar{z}_1)U_2(z_2, \bar{z}_2) \int d^2y \int d^2z_4 \log|\frac{(y-z_3)(z_4-z_3)}{y-z_4}|\{Q, [\bar{Q}, V_3(y, \bar{y})]\} \tag{3.6}$$

$$V_4(z_4, \bar{z}_4) \int d^2z_5 V_5(z_5, \bar{z}_5) ... \int d^2z_N V_N(z_N, \bar{z}_N)\rangle$$

where we have used that

$$\frac{1}{2\pi}\partial_y\bar\partial_{\bar y}\log|\frac{(y-z_3)(z_4-z_3)}{y-z_4}|=\delta^2(y-z_3)-\delta^2(y-z_4) \tag{3.7}$$

and that the contribution from $V_3(z_4,\bar z_4)$ can be ignored using the cancelled propagator argument. Note that the argument of the logarithm has been chosen such that the logarithm is non-singular as $y\to\infty$. Pulling Q and $\bar Q$ off of $V_3(y,\bar y)$, the only contribution comes when they circle $V_4(z_4,\bar z_4)$ to give

$$\mathcal{A}=\frac{1}{2\pi}\langle U_1(z_1,\bar z_1)U_2(z_2,\bar z_2)\int d^2yV_3(y,\bar y)\int d^2z_4\log|\frac{(y-z_3)(z_4-z_3)}{y-z_4}| \tag{3.8}$$

$$\{Q,[\bar Q,V_4(z_4,\bar z_4)]\}\int d^2z_5V_5(z_5,\bar z_5)...\int d^2z_NV_N(z_N,\bar z_N)\rangle$$

$$=\langle U_1(z_1,\bar z_1)U_2(z_2,\bar z_2)\int d^2yV_3(y,\bar y)V_4(z_3,\bar z_3)\int d^2z_5V_5(z_5,\bar z_5)...\int d^2z_NV_N(z_N,\bar z_N)\rangle \tag{3.9}$$

which is the closed tree amplitude prescription with a different choice of unintegrated vertex operators.

It will now be proven that for amplitudes involving an arbitrary number of massless bosons and up to four massless fermions, the prescription given by (2.7), (2.8) and (2.9) coincides with the standard RNS prescription of [2]. This will first be proven for amplitudes involving four fermions, then for amplitudes involving two fermions, and finally, for amplitudes involving zero fermions.

4. Equivalence for Amplitudes involving Four Fermions

Because of the cyclic symmetry proven in the previous section, one is free to choose three of the four fermion vertex operators to be unintegrated. With this choice, the amplitude prescription of (2.9) is

$$\mathcal{A}=\langle\xi_1^\alpha U^F_\alpha(z_1)\xi_2^\beta U^F_\beta(z_2)\xi_3^\gamma U^F_\gamma(z_3) \tag{4.1}$$

$$\int dz_4\xi_4^\delta V^F_\delta(z_4)\int dz_5a_5^mV^B_m(z_5)...\int dz_Na_N^nV^B_n(z_N)\rangle$$

where ξ^α and a^m are the polarizations and $(U^F_\alpha,V^F_\alpha,V^B_m)$ are defined in (2.7) and (2.8). Since U^F_α has a minimum of two θ's and since $\mathcal{A}$ requires precisely five θ zero modes to be non-vanishing, the only terms in $(U^F_\alpha,V^F_\alpha,V^B_m)$ which contribute are

$$\mathcal{A}=-\frac{1}{27}\langle\xi_1^\alpha f_\alpha(z_1)\xi_2^\beta f_\beta(z_2)\xi_3^\gamma f_\gamma(z_3)\int dz_4\xi_4^\delta p_\delta(z_4) \tag{4.2}$$

$$\int dz_5 a_5^m(\partial x_m(z_5) - ik_5^p M_{mp}(z_5))... \int dz_N a_N^n(\partial x_n(z_N) - ik_N^q M_{nq}(z_N))e^{i\sum_{r=1}^N k_r\cdot x(z_r)}\rangle$$

where $f_\alpha \equiv (\lambda\gamma^m\theta)(\gamma_m\theta)_\alpha$ and $M_{mn} \equiv N_{mn} - \frac{1}{2}(p\gamma_{mn}\theta)$.

The amplitude prescription of (4.2) will now be shown to coincide with the RNS prescription of [2] with the four fermion vertex operators in the $-\frac{1}{2}$ picture.[5] Choosing three of the fermion vertex operators to be unintegrated,

$$\mathcal{A}_{RNS} = -\langle \xi_1^\alpha c e^{-\frac{\phi}{2}}\Sigma_\alpha(z_1)\xi_2^\beta c e^{-\frac{\phi}{2}}\Sigma_\beta(z_2)\xi_3^\gamma c e^{-\frac{\phi}{2}}\Sigma_\gamma(z_3)\int dz_4 \xi_4^\delta e^{-\frac{\phi}{2}}\Sigma_\delta(z_4) \tag{4.3}$$

$$\int dz_5 a_5^m(\partial x_m(z_5) - ik_5^p\psi_m\psi_p(z_5))... \int dz_N a_N^n(\partial x_n(z_N) - ik_N^q\psi_n\psi_q(z_N))e^{i\sum_{r=1}^N k_r\cdot x(z_r)}\rangle$$

where Σ_α is the RNS spin field, $-\xi^\alpha c e^{-\frac{\phi}{2}}\Sigma_\alpha$ is the unintegrated fermion vertex operator, and $\xi^\alpha e^{-\frac{\phi}{2}}\Sigma_\alpha = \{b, -\xi^\alpha c e^{-\frac{\phi}{2}}\Sigma_\alpha\}$ is the integrated fermion vertex operator.

The correlation function of x^m is clearly equivalent in $\mathcal{A}$ and $\mathcal{A}_{RNS}$ of (4.2) and (4.3). So to show $\mathcal{A} = \mathcal{A}_{RNS}$, one only needs to show that

$$\frac{1}{27}\langle f_\alpha(z_1)f_\beta(z_2)f_\gamma(z_3)p_\delta(z_4)M_{mp}(z_5)...M_{nq}(z_N)\rangle = \tag{4.4}$$

$$\langle c e^{-\frac{\phi}{2}}\Sigma_\alpha(z_1)c e^{-\frac{\phi}{2}}\Sigma_\beta(z_2)c e^{-\frac{\phi}{2}}\Sigma_\gamma(z_3)e^{-\frac{\phi}{2}}\Sigma_\delta(z_4)\psi_m\psi_p(z_5)...\psi_n\psi_q(z_N)\rangle.$$

To prove (4.4), first note that (2.1) implies that [1]

$$M_{kl}(y)M_{mn}(z) \to \frac{\eta_{m[l}M_{k]n}(z) - \eta_{n[l}M_{k]m}(z)}{y-z} + \frac{\eta_{kn}\eta_{lm} - \eta_{km}\eta_{ln}}{(y-z)^2}, \tag{4.5}$$

which coincides with the OPE of $\psi_k\psi_l(y)$ with $\psi_m\psi_n(z)$. Furthermore,

$$M_{mn}(y)f_\alpha(z) \to \frac{(\gamma_{mn})_\alpha{}^\beta f_\beta(z)}{2(y-z)}, \quad M_{mn}(y)p_\alpha(z) \to \frac{(\gamma_{mn})_\alpha{}^\beta p_\beta(z)}{2(y-z)}, \tag{4.6}$$

reproduces the OPE of $\psi_m\psi_n(y)$ with $\Sigma_\alpha(z)$. Since the dependence of $\mathcal{A}$ and $\mathcal{A}_{RNS}$ on $z_5...z_N$ is completely determined by these OPE's, we have shown that $\mathcal{A} = \mathcal{A}_{RNS}$ if

$$\frac{1}{27}\langle f_\alpha(z_1)f_\beta(z_2)f_\gamma(z_3)p_\delta(z_4)\rangle = \langle c e^{-\frac{\phi}{2}}\Sigma_\alpha(z_1)c e^{-\frac{\phi}{2}}\Sigma_\beta(z_2)c e^{-\frac{\phi}{2}}\Sigma_\gamma(z_3)e^{-\frac{\phi}{2}}\Sigma_\delta(z_4)\rangle. \tag{4.7}$$

Using the OPE's of [2], the right-hand side of (4.7) is easily evaluated to be

$$\frac{\gamma^m_{\alpha\delta}\gamma_{m\,\beta\gamma}}{z_1 - z_4} + \frac{\gamma^m_{\beta\delta}\gamma_{m\,\gamma\alpha}}{z_2 - z_4} + \frac{\gamma^m_{\gamma\delta}\gamma_{m\,\alpha\beta}}{z_3 - z_4}. \tag{4.8}$$

[5] Comparison of the two prescriptions is complicated for amplitudes involving more than four fermions since such amplitudes require fermion vertex operators in the $+\frac{1}{2}$ picture.

The left-hand side of (4.7) can also be evaluated by analyzing the poles of $p_\delta(z_4)$. For example, as $z_4 \to z_1$, the left-hand side has a pole whose residue is

$$\frac{1}{27}\langle\ [(\gamma^m_{\alpha\delta}(\lambda\gamma_m\theta)(z_1) - (\gamma_m\lambda)_\delta(\gamma^m\theta)_\alpha(z_1)]\ (\lambda\gamma^n\theta)(\gamma_n\theta)_\beta(z_2)\ (\lambda\gamma^p\theta)(\gamma_p\theta)_\gamma(z_3)\ \rangle. \quad (4.9)$$

To simplify the evaluation of (4.9), use the fact that

$$-(\gamma_m\lambda)_\delta(\gamma^m\theta)_\alpha = \frac{1}{2}[(\gamma_m\lambda)_\alpha(\gamma^m\theta)_\delta - (\gamma_m\lambda)_\delta(\gamma^m\theta)_\alpha] + \frac{1}{2}\gamma^m_{\alpha\delta}(\theta\gamma_m\lambda) \quad (4.10)$$

$$= Q[\frac{1}{2}(\gamma_m\theta)_\alpha(\gamma^m\theta)_\delta] + \frac{1}{2}\gamma^m_{\alpha\delta}(\theta\gamma_m\lambda)$$

where $Q = \oint dz\lambda^a d_a$. Since Q anti-commutes with the vertex operators at z_2 and z_3 and since $\langle Q(Y)\rangle = 0$ for any Y [1], the term $-(\gamma_m\lambda)_\delta(\gamma^m\theta)_\alpha$ in (4.9) can be replaced with $\frac{1}{2}\gamma^m_{\alpha\delta}(\theta\gamma_m\lambda)$. So using the zero mode correlation function defined in (2.11), (4.9) is equal to $\frac{1}{18}\gamma^m_{\alpha\delta}H_{m\,\beta\gamma}$ where

$$H_{m\,\beta\gamma} = \langle\ (\lambda\gamma_m\theta)(z_1)\ (\lambda\gamma^n\theta)(\gamma_n\theta)_\beta(z_2)\ (\lambda\gamma^p\theta)(\gamma_p\theta)_\gamma(z_3)\ \rangle = 18\gamma_{m\,\beta\gamma}. \quad (4.11)$$

To prove (4.11), we have used that (2.11) is Lorentz-invariant so $H_{m\,\beta\gamma}$ must be proportional to $\gamma_{m\,\beta\gamma}$. To find the proportionality constant, we have used from (2.10) that

$$\gamma^{m\,\beta\gamma}H_{m\,\beta\gamma} = \langle(\lambda\gamma_m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 2880.$$

So the residue of the $\frac{1}{z_1-z_4}$ pole in (4.9) is $\gamma^m_{\alpha\delta}\gamma_{m\,\beta\gamma}$ which agrees with the residue in (4.8). Similarly, one can show that the residues of the $\frac{1}{z_2-z_4}$ and $\frac{1}{z_3-z_4}$ poles agree in the two expressions so we have proven that $\mathcal{A} = \mathcal{A}_{RNS}$ for amplitudes involving four fermions.

5. Equivalence for Amplitudes involving Two Fermions

The proof of equivalence for amplitudes involving two fermions closely resembles the proof for amplitudes involving four fermions. Choosing two fermion vertex operators and one boson vertex operator to be unintegrated, the amplitude prescription in the covariant formalism is

$$\mathcal{A} = \langle\xi_1^\alpha U^F_\alpha(z_1)\xi_2^\beta U^F_\beta(z_2)a_3^m U^B_m(z_3)\int dz_4 a_4^n V^B_n(z_4)...\int dz_N a_N^p V^B_p(z_N)\rangle \quad (5.1)$$

where ξ^α and a^m are the polarizations and $(U^F_\alpha, U^B_m, V^B_m)$ are defined in (2.7) and (2.8). Since U^F_α has a minimum of two θ's, U^B_m has a minimum of one θ, and $\mathcal{A}$ requires precisely five θ zero modes to be non-vanishing, the only terms in $(U^F_\alpha, U^B_m, V^B_m)$ which contribute are

$$\mathcal{A} = \frac{1}{18}\langle \xi_1^\alpha f_\alpha(z_1)\xi_2^\beta f_\beta(z_2) a_3^m b_m(z_3) \tag{5.2}$$

$$\int dz_4 a_4^n(\partial x_n(z_4) - ik_4^q M_{nq}(z_4))...\int dz_N a_N^p(\partial x_p(z_N) - ik_N^r M_{pr}(z_N))e^{i\sum_{r=1}^N k_r\cdot x(z_r)}\rangle$$

where $f \equiv (\lambda\gamma^m\theta)(\gamma_m\theta)_\alpha$, $b_m \equiv \lambda\gamma_m\theta$, and $M_{mn} \equiv N_{mn} - \frac{1}{2}(p\gamma_{mn}\theta)$.

The amplitude prescription of (5.2) will now be shown to coincide with the RNS prescription of [2],

$$\mathcal{A}_{RNS} = \langle \xi_1^\alpha c e^{-\frac{\phi}{2}}\Sigma_\alpha(z_1)\xi_2^\beta c e^{-\frac{\phi}{2}}\Sigma_\beta(z_2) a_3^m c e^{-\phi}\psi_m(z_3) \tag{5.3}$$

$$\int dz_4 a_4^n(\partial x_n(z_4) - ik_4^q\psi_n\psi_q(z_4))...\int dz_N a_N^p(\partial x_p(z_N) - ik_N^r\psi_p\psi_r(z_N))e^{i\sum_{r=1}^N k_r\cdot x(z_r)}\rangle$$

where the fermion vertex operators are in the $-\frac{1}{2}$ picture and the unintegrated boson vertex operator is in the -1 picture.

As before, the correlation function of x^m is equivalent in $\mathcal{A}$ and $\mathcal{A}_{RNS}$ of (5.2) and (5.3). Furthermore,

$$M_{mn}(y)f_\alpha(z) \to \frac{(\gamma_{mn})_\alpha{}^\beta f_\beta(z)}{2(y-z)}, \quad M_{mn}(y)b_p(z) \to \frac{\eta_{np}b_m(z) - \eta_{mp}b_n(z)}{y-z}, \tag{5.4}$$

reproduces the OPE of $\psi_m\psi_n(y)$ with $\Sigma_\alpha(z)$ and with $\psi_p(z)$. So using the arguments of the previous section, $\mathcal{A} = \mathcal{A}_{RNS}$ if

$$\frac{1}{18}\langle f_\alpha(z_1)f_\beta(z_2)b_m(z_3)\rangle = \langle c e^{-\frac{\phi}{2}}\Sigma_\alpha(z_1) c e^{-\frac{\phi}{2}}\Sigma_\beta(z_2) c e^{-\phi}\psi_m(z_3)\rangle. \tag{5.5}$$

Using the RNS OPE's of [2], the right-hand side of (5.5) is easily evaluated to be $\gamma_{m\,\alpha\beta}$. The left-hand side of (4.7) is

$$\frac{1}{18}\langle\ (\lambda\gamma^n\theta)(\gamma_n\theta)_\alpha(z_1)\ (\lambda\gamma^p\theta)(\gamma_p\theta)_\beta(z_2)\ (\lambda\gamma_m\theta)(z_3)\rangle = \frac{1}{18}H_{m\,\alpha\beta} = \gamma_{m\,\alpha\beta} \tag{5.6}$$

from (4.11). So we have proven that $\mathcal{A} = \mathcal{A}_{RNS}$ for amplitudes involving two fermions.

6. Equivalence for Amplitudes involving Zero Fermions

The equivalence of amplitudes involving zero fermions will now be proven using space-time supersymmetry to relate these amplitudes with amplitudes involving two fermions. This will be made explicit uusing the supersymmetry transformations of the covariant and RNS massless vertex operators.

First, note that the supersymmetry generator of (2.5) exchanges the massless boson and fermion vertex operators of (2.7) and (2.8) in the following manner:

$$\{q_\alpha, U_m^B\} = \frac{i}{2} k^n (\gamma_{mn})_\alpha{}^\beta U_\beta^F + Q(\Omega_{m\alpha}), \quad [q_\alpha, U_\beta^F] = \gamma_{\alpha\beta}^m U_m^B + Q(\Sigma_{\alpha\beta}), \tag{6.1}$$

$$[q_\alpha, V_m^B] = \frac{i}{2} k^n (\gamma_{mn})_\alpha{}^\beta V_\beta^F - \partial(\Omega_{m\alpha}), \quad \{q_\alpha, V_\beta^F\} = \gamma_{\alpha\beta}^m V_m^B + \partial(\Sigma_{\alpha\beta}),$$

for some $\Omega_{m\alpha}$ and $\Sigma_{\alpha\beta}$. (6.1) can be derived either by explicit computation or by using the on-shell supersymmetry transformations of the super-Yang-Mills component fields. The dependence on $\Omega_{m\alpha}$ and $\Sigma_{\alpha\beta}$ comes from the fact that supersymmetry transformations do not commute with the gauge choice of (2.6).

The covariant amplitude prescription for the scattering of N massless bosons is

$$\mathcal{A} = \langle a_1^m U_m^B(z_1) a_2^n U_n^B(z_2) a_3^p U_p^B(z_3) \int dz_4 a_4^q V_q^B(z_4) ... \int dz_N a_N^r V_r^B(z_N) \rangle, \tag{6.2}$$

which can be written using (6.1) and BRST-invariance of the correlation function as

$$\mathcal{A} = \frac{1}{16} \langle a_1^m \gamma_m^{\alpha\beta} [q_\alpha, U_\beta^F(z_1)] a_2^n U_n^B(z_2) a_3^p U_p^B(z_3) \int dz_4 a_4^q V_q^B(z_4) ... \int dz_N a_N^r V_r^B(z_N) \rangle. \tag{6.3}$$

Since the correlation function preserves supersymmetry as was shown in [1], q_α can be pulled off of $U_\beta^F(z_1)$ until it circles any of the other boson vertex operators.

For example, when q_α circles $U_n^B(z_2)$, one gets the term

$$-\frac{1}{16} \langle a_1^m \gamma_m^{\alpha\beta} U_\beta^F(z_1) a_2^n \{q_\alpha, U_n^B(z_2)\} a_3^p U_p^B(z_3) \int dz_4 a_4^q V_q^B(z_4) ... \int dz_N a_N^r V_r^B(z_N) \rangle \tag{6.4}$$

$$= -\frac{i}{32} \langle a_1^m \gamma_m^{\alpha\beta} U_\beta^F(z_1) a_2^n k_2^s (\gamma_{ns})_\alpha{}^\delta U_\delta^F(z_2) a_3^p U_p^B(z_3) \int dz_4 a_4^q V_q^B(z_4) ... \int dz_N a_N^r V_r^B(z_N) \rangle.$$

But using the results of section 5, this is equal to the analagous RNS correlation function where U_α^F is replaced with the picture $-\frac{1}{2}$ fermion vertex operator $-ce^{-\frac{\phi}{2}} \Sigma_\beta e^{ik\cdot x}$, U_m^B is replaced with the picture -1 boson vertex operator $ce^{-\phi}\psi_p e^{ik\cdot x}$, and V_m^B is replaced with $(\partial x_m - ik^n \psi_m \psi_n) e^{ik\cdot x}$.

Similarly, when q_α circles $V_q^B(z_4)$, one gets the term

$$-\frac{i}{32}\langle a_1^m \gamma_m^{\alpha\beta} U_\beta^F(z_1) a_2^n U_n^B(z_2) a_3^p U_p^B(z_3) \int dz_4 a_4^q k_4^s (\gamma_{qs})_\alpha{}^\delta V_\delta^F(z_4) \tag{6.5}$$

$$\int dz_N a_5^r V_r^B(z_5) ... \int dz_N a_N^s V_s^B(z_N)\rangle.$$

To relate (6.5) to an analogous RNS expression, one first uses the results of section 3 to exchange $U_p^B(z_3)\int dz_4 V_\delta^F(z_4)$ for $\int dy V_p^B(y)\ U_\delta^F(z_3)$. One can then use the results of section 5 to relate (6.5) to an analogous RNS expression as was done for (6.4).

So $\mathcal{A}$ of (6.2) is equal to a sum of RNS correlation functions involving $N-2$ massless boson vertex operators and two massless fermion vertex operators. It will now be shown that this sum of RNS correlation functions is related by supersymmetry to the RNS prescription for the scattering of N massless bosons:

$$\mathcal{A}_{RNS} = \langle \{Q, \xi(z_0)\} a_1^m c e^{-\phi}\psi_m(z_1) a_2^n c e^{-\phi}\psi_n(z_2) a_3^p c e^{-\phi}\psi_p(z_3) \tag{6.6}$$

$$\int dz_4 a_4^q (\partial x_q(z_5) - i k_5^s \psi_q \psi_s(z_4)) ... \int dz_N a_5^r (\partial x_r(z_N) - i k_N^t \psi_r \psi_t(z_N)) e^{i\sum_{r=1}^N k_r\cdot x(z_r)}\rangle,$$

where $\{Q, \xi(z_0)\}$ is the picture-raising operator and z_0 is arbitrary. To prove $\mathcal{A} = \mathcal{A}_{RNS}$, first write

$$\mathcal{A}_{RNS} = \frac{1}{16}\langle \{Q, \xi(z_0)\} a_1^m \gamma_m^{\alpha\beta} [q_\alpha^{RNS}, -c e^{-\frac{\phi}{2}}\Sigma_\beta(z_1)] a_2^n c e^{-\phi}\psi_n(z_2) a_3^p c e^{-\phi}\psi_p(z_3) \tag{6.7}$$

$$\int dz_4 a_4^q (\partial x_q(z_4) - i k_4^s \psi_q \psi_s(z_4)) ... \int dz_N a_5^r (\partial x_r(z_N) - i k_N^t \psi_r \psi_t(z_N)) e^{i\sum_{r=1}^N k_r\cdot x(z_r)}\rangle,$$

where $q_\alpha^{RNS} = \oint dz e^{-\frac{\phi}{2}}\Sigma_\alpha$ is the RNS spacetime-supersymmetry generator in the $-\frac{1}{2}$ picture[2]. Pulling q_α^{RNS} off of $-c e^{-\frac{\phi}{2}}\Sigma_\beta(z_1)$ until it circles the other vertex operators, one recovers precisely the same terms as found earlier.

For example, if q_α^{RNS} circles $c e^{-\phi}\psi_n(z_2)$, one obtains the term $\gamma_{n\,\alpha\beta} c e^{-\frac{3\phi}{2}}\Sigma^\beta(z_2)$. Choosing $z_0 = z_2$, one gets the picture-raised version of this term which is $\frac{i}{2}k_2^s(\gamma_{ns})_\alpha{}^\delta$ $c e^{-\frac{\phi}{2}}\Sigma_\delta(z_2)$. Comparing expressions, one sees that this is precisely the RNS version of (6.4). Similarly, if q_α^{RNS} circles $\partial x_q(z_4) - i k_4^s \psi_q\psi_s(z_4)$, one obtains $\frac{i}{2}k_4^s(\gamma_{qs})_\alpha{}^\beta e^{-\frac{\phi}{2}}\Sigma_\beta(z_4)$. Choosing $z_0 = z_3$ to convert $c e^{-\phi}\psi_p(z_3)$ to $c(\partial x_p(z_3) - i k_3^u \psi_p\psi_u(z_3))$ and using cyclic symmetry to exchange the integrated and unintegrated vertex operators at z_3 and z_4, one recovers the RNS version of (6.5). So we have proven the equivalence of amplitudes involving zero fermions.

Acknowledgements: NB would like to thank Warren Siegel and Edward Witten for useful discussions and CNPq grant 300256/94-9 for partial financial support. BCV would like to thank Osvaldo Chandia for useful discussions and Fapesp grant 98/15374-1 for financial support.

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